International Journal of Theoretical Physics, Vol. 10, No. 3 (1974), pp. 179–187

Some Zero Rest Mass Test Fields in General Relativity

A. HELD

Institut für Theoretische Physik, Universität Bern

and B. H. VOORHEES

Department of Mathematics, University of Alberta, Edmonton 7, Canada

Received: 29 *September* 1973

Abstract

Zero rest mass test fields of pure algebraic type are defined and studied via the newly developed GHP formalism. The field equations are written explicitly and an immediate generalisation of Robinson's theorem is obtained. The form for the general zero rest mass test field of pure type is determined in terms of the tetrad components of the background Weyl spinor and an additional gauge dependent function which can be thought of as representing a test neutrino field.

1. *Introduction*

It is well known that arbitrary vacuum space-times do no necessarily admit specialized zero rest mass (ZRM) test fields. The first and most widely known results of this nature are contained in Robinson's theorem (1961) and the Goldberg-Sachs theorem (1962). By Robinson's theorem a vacuum space-time admits a null Maxwell test field if and only if it contains a shear-free congruence of null geodesics which will then be tangent to the principal null direction (PND) of the field. Therefore, by the Goldberg-Sachs theorem the background space-time is algebraically special with the repeated PND of the Weyl tensor also lying along this congruence.

Recently, interest has been directed towards extension of this result to ZRM fields more general than Maxwell (Bell & Szekeres, 1972; Collinson, 1973). The present paper continues such efforts by providing partial answers to the following two questions:

(1) If $\Phi_{AB\ldots D}$ is to be a spin s ZRM test field of pure type (see below) residing in a background space-time with Weyl spinor Ψ_{ABCD}, what, if any, restrictions must be placed on Ψ_{ABCD}?

(2) Does there exist any simple relationship between the spinors $\Phi_{AB\ldots D}$ and Ψ_{ABCD} of question (1)?

Every spin s ZRM field may be represented by a totally symmetric spinor $\chi_{AB\ldots D}$ having $2s$ indices and satisfying the spinor equation (Penrose, 1965)

$$\nabla^{A}{}_{A'}\chi_{AB\ldots D} = 0 \tag{1.1}$$

If (o^A, ι^A) is a spinor dyad there is an associated decomposition (Pirani, 1964)

$$\chi_{AB\ldots D} = \sum_{t=0}^{2s} \Phi^{(t)}_{A\ldots BC\ldots D} \tag{1.2}$$

where

$$\Phi^{(t)}_{A\ldots BC\ldots D} = \Phi^{(t)} o_{(A} \ldots o_B \iota_C \ldots \iota_{D)} \tag{1.3}$$

and

$$\Phi^{(2s-t)} = \chi_{AB\ldots CD}\, o^A \ldots o^B \iota^C \ldots \iota^D \tag{1.4}$$

the spinor o^A appearing t times and the spinor ι^A $2s - t$ times.

Spinors of the form (1.3) will be said to be of pure algebraic type $[t;r]$ where t represents the number of o^A's and r the number of ι^A's in the canonical decomposition. Every ZRM field may be decomposed into spinor fields of pure type, none of which, in general, are solutions of the ZRM field equation. It is obvious that a totally symmetric spinor field will be of pure type if and only if it possesses not more than two PND's.

It is a consequence of the dyad form of the general ZRM field equations (1.1) that the field $\Phi^{(t)}[t;2s-t]$ (for fixed t) in (1.2) will be a solution of (1.1) if

$$\Phi^{(t+2)} = \Phi^{(t+1)} = \Phi^{(t-1)} = \Phi^{(t-2)} = 0$$

In the present paper only solutions of (1.1), which are themselves of pure type, will be considered. Since such fields will have at most two PND's they are ideally suited for analysis via the recently developed formalism of Geroch, Held & Penrose (1973). A brief review of this formalism, which is an adaptation of the well-known Newman–Penrose formalism to space-time containing a preferred pair of null directions, is given in Section 2.

2. *The GHP Formalism*

Essentially this formalism is a restriction of the Newman–Penrose formalism (1962) to an algebra of objects with specific behavior under the gauge transformation

$$\begin{aligned} o^A &\to \lambda o^A \\ \iota^A &\to \lambda^{-1}\iota^A \end{aligned} \tag{2.1}$$

of the base dyad. If η is an element of this algebra it will behave under such a transformation as

$$\eta \to \lambda^p \bar{\lambda}^{-q} \eta \tag{2.2}$$

and is said to be of GHP type (p, q). The algebra is completed by the introduction of a set of differential operators which map objects of definite type into objects of definite type. These are designated þ (thorn), þ′, ð (edth), ð′ and are defined by

$$\begin{aligned} þ\,\eta &= (D - p\epsilon - q\bar{\epsilon})\eta \\ þ'\eta &= (\Delta - p\gamma - q\bar{\gamma})\eta \\ ð\,\eta &= (\delta - p\beta - q\bar{\alpha})\eta \\ ð'\eta &= (\bar{\delta} - p\alpha - q\bar{\beta})\eta \end{aligned} \tag{2.3}$$

where η has GHP type (p, q).

Eight of the Newman–Penrose spin coefficients are found to be elements of the algebra. The remaining four, together with the differential operators D, Δ, δ, $\bar{\delta}$, are used to form the derivative operators of (2.3). The Bianchi identities and twelve of the eighteen field equations are easily rewritten as equations of this algebra. The remaining six field equations are discarded, their information content reappearing in the commutators of the operators in (2.3).

A further simplification is obtained by denoting the symmetry operation

$$o^A \to i\iota^A; \qquad \iota^A \to io^A \tag{2.4}$$

by ′ (prime) and labeling the spin coefficients accordingly. Thus the spin coefficients belonging to the algebra are ρ, κ, σ, τ and ρ', κ', σ', ρ' (i.e., $-\mu$, $-\nu$, $-\lambda$, $-\pi$). The GHP equations required for the purpose of this paper are ($\Psi_0 = \Psi_1 = \sigma = \kappa = 0$):

The vacuum Bianchi identities:

$$þ\Psi_2 = 3\rho\Psi_2 \tag{2.5a}$$

$$ð\Psi_2 = 3\tau\Psi_2 \tag{2.5b}$$

$$þ'\Psi_2 - ð\Psi_3 = -2\tau\Psi_3 + 3\rho'\Psi_2 \tag{2.5c}$$

$$þ\Psi_3 - ð'\Psi_2 = -3\tau'\Psi_2 + 2\rho\Psi_3 \tag{2.5d}$$

$$þ'\Psi_3 - ð\Psi_4 = -\tau\Psi_4 + 4\rho'\Psi_3 \tag{2.5e}$$

$$þ\Psi_4 - ð'\Psi_3 = 3\sigma'\Psi_2 - 4\tau'\Psi_3 + \rho\Psi_4 \tag{2.5f}$$

and the field equations:

$$þ'\rho - ð'\tau = \bar{\rho}'\rho - \tau\bar{\tau} - \Psi_2 \tag{2.6a}$$

$$þ\rho' - ð\tau' = \rho'\bar{\rho} - \tau'\bar{\tau}' - \Psi_2 \tag{2.6b}$$

The commutation relations used are:

$$[Þ,Þ']\eta = [(\bar{\tau} - \tau')ð + (\tau - \bar{\tau}')ð' + p(\tau\tau' - \Psi_2) + q(\bar{\tau}\bar{\tau}' - \bar{\Psi}_2)]\eta \quad (2.7a)$$

$$[ð,ð']\eta = [(\bar{\rho}' - \rho')Þ + (\rho - \bar{\rho})Þ' + p(\rho\rho' + \Psi_2) - q(\bar{\rho}\bar{\rho}' + \bar{\Psi}_2)]\eta \quad (2.7b)$$

$$[Þ',ð']\eta = [\bar{\rho}'ð' + \sigma'ð - \bar{\tau}Þ' - \kappa'Þ + p(\rho\kappa' - \tau\sigma' + \Psi_3) - q\bar{\rho}'\bar{\tau}]\eta \quad (2.7c)$$

$$[Þ,ð]\eta = [\bar{\rho}ð - \bar{\tau}'Þ - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}')]\eta \quad (2.7d)$$

3. *Zero Rest Mass Test Fields*

If $\Phi_{A\ldots BC\ldots D} = \Phi[t;r]$ is a ZRM test field the scalar Φ is of GHP type $(r - t, 0)$ and in the canonical dyad equations (1.1) become:

$$t[Þ\Phi - (r+1)\rho\Phi] = 0 \quad (3.1a)$$

$$t[ð\Phi - (r+1)\tau\Phi] = 0 \quad (3.1b)$$

$$r[Þ'\Phi - (t+1)\rho'\Phi] = 0 \quad (3.1c)$$

$$r[ð'\Phi - (t+1)\tau'\Phi] = 0 \quad (3.1d)$$

$$t(t-1)\kappa = t(t-1)\sigma = 0 \quad (3.1e)$$

$$r(r-1)\kappa' = r(r-1)\sigma' = 0 \quad (3.1f)$$

It is obvious from (3.1e, f) that if either $t > 1$ or $r > 1$ then constraints on the background space-time are implied via the Goldberg–Sachs theorem, the case $[t;r] = [2;0]$ or $[0;2]$ giving a statement of the first part of Robinson's theorem.

Suppose $t > 1$. Then by (3.1e) the space-time must possess a shear free congruence of null geodesics ($\kappa = \sigma = 0$) to which the vector $l^a = o^A\bar{o}^{A'}$ is tangent and hence must be algebraically special with o^A a repeated principal spinor of the Weyl spinor ψ_{ABCD}. There are now three cases to consider, these being $r > 1$, $r = 1$ and $r = 0$.

If $r > 1$ then by (3.1f) $\kappa' = \sigma' = 0$, hence $n^a = \iota^A\bar{\iota}^{A'}$ is geodetic and shear free so that ψ_{ABCD} is Petrov type D with repeated principal spinors proportional to o^A and ι^A.

If the substitution

$$\Phi = \Theta\Psi_2^{(r+1)/3} \quad (3.2)$$

is made then use of the Bianchi identities (2.5) to (2.8) reduces (3.1a–d) to

$$Þ\Theta = 0 \quad (3.3a)$$

$$ð\Theta = 0 \quad (3.3b)$$

$$Þ'\Theta = (t - r)\rho'\Theta \quad (3.4a)$$

$$ð'\Theta = (t - r)\tau'\Theta \quad (3.4b)$$

Application of the commutators $[\text{Þ},\text{Þ}']$ and $[\eth,\eth']$ to Θ coupled with (3.3) and (3.4) yields the integrability conditions

$$(t-r)(\text{Þ}\rho' + \tau'\bar{\tau} - \Psi_2) = 0$$

and

$$(t-r)(\eth\tau' + \rho'\bar{\rho} + \Psi_2) = 0 \tag{3.5}$$

Comparison of these with (2.6b) now leads to the requirement

$$(t-r)\Psi_2 = 0 \tag{3.6}$$

if the space-time is not flat then $r = t$. Returning to (3.3) and (3.4) with $t = r$ we observe that Θ must be constant (cf. Section 4).

If $r = 1$ suppose that in addition Ψ_2 is not zero. Then there exists a null rotation

$$\begin{aligned} o^A &\to o^A \\ \iota^A &\to \iota^A + a o^A \equiv \mu^A \end{aligned} \tag{3.7}$$

such that in the dyad (o^A, μ^A), $\Psi_3 = 0$. In this dyad (1.1) becomes

$$\nabla^A_{A'}\{\Phi o_{(A} \ldots o_B \mu_{C)} + a\Phi o_A \ldots o_B o_C\} = 0 \tag{3.8}$$

there being $t + 1$ unprimed subscripts. The projections of this equation onto the dyad (o^A, μ^A) yields the GHP type equations

$$\text{Þ}\Phi = 2\rho\Phi \tag{3.9a}$$

$$\eth\Phi = 2\tau\Phi \tag{3.9b}$$

$$\eth'\Phi - (t+1)\tau'\Phi + (t+1)(\text{Þ}a\Phi - a\rho\Phi) = 0 \tag{3.10a}$$

$$\text{Þ}'\Phi - (t+1)\rho'\Phi + (t+1)(\eth a\Phi - a\tau\Phi) = 0 \tag{3.10b}$$

Setting

$$\phi = \Psi_2^{-1/3}\Phi$$

and

$$\theta = (t+1)^{-1}a\phi \tag{3.11}$$

reduces these equations to

$$\text{Þ}\phi = \rho\phi \tag{3.9a$'$}$$

$$\eth\phi = \tau\phi \tag{3.9b$'$}$$

$$\text{Þ}\theta = \eth'\phi - t\tau'\phi \tag{3.10a$'$}$$

$$\eth\theta = \text{Þ}'\phi - t\rho'\phi \tag{3.10b$'$}$$

Now consider the commutator $[\text{Þ},\eth]$ acting on θ. Use of (2.7a) to (2.7d) in a manner similar to the derivation of (3.6) gives the integrability condition

$$(t-1)\phi\Psi_2 = 0 \tag{3.12}$$

Since $t > 1$ and $\phi \neq 0$ this is satisfied if and only if $\Psi_2 = 0$, which contradicts the original assumption. Hence Ψ_2 must be zero.

These results may be stated as a theorem which generalizes recent results of Bell & Szekeres (1972):

Theorem. Let $\Phi_{A \ldots BC \ldots D}$ be a ZRM test field of pure algebraic type $[t; r]$ $(t > 1)$ which lives in a non-flat vacuum background. Then,

(a) $r = 0$ and Ψ_{ABCD} is algebraically special; or
(b) $r = 1$ and Ψ_{ABCD} is type III or type N; or
(c) $r > 1$, Ψ_{ABCD} is type D and $r = t$.

Corollary. Apart from type $[t; 0]$ test fields which are admitted by all algebraically special space-times the admissable ZRM test fields of pure type $[t; r]$ $(t > 1)$ are:

(a) in a type N space-time—$[t; 1]$,
(b) in a type III space-time—$[t; 1]$ (although there are special restrictions if ι^A is parallel to the remaining PND of Ψ_{ABCD} (see section 4)),
(c) in a type D space-time—$[t; t]$,
(d) in a type II space-time—none.

4. *Some Special Solutions*

With a single exception, all of the possibilities considered in Section 3 admit only specific solutions. These are enumerated in Table 1, $(t > 1)$.

Here A is an arbitrary constant and ζ is the general solution of the pair of equations

$$þ\zeta = 0$$
$$ð\zeta = 0 \tag{4.1}$$

(A corresponding table may be constructed for solutions of types $[0; t]$ and $[1; t]$ with ζ then satisfying $þ'\zeta = ð'\zeta = 0$.)

Equations (4.1) admit two functionally independent solutions the existence of which is guaranteed by a lemma proved in the Appendix.

There is one special case, that of a $[t; 1]$ test field in a background type III space-time. Now the problem splits into two parts according as to whether the spinor ι^A is parallel to the non-repeated PND of Ψ_{ABCD} or not. If it is parallel

TABLE 1. The form of Φ in algebraically special backgrounds

	Algebraic type of background			
Test field type	II	D	III	N
$[t; 0]$	$\zeta^t \Psi_2^{1/3}$†	$\zeta^t \Psi_2^{1/3}$	$\zeta^{t-1} \Psi_3^{1/2}$	$\zeta^{t-4} \Psi_4$
$[t; 1]$	—	—	see below	$\zeta^{t-9} \Psi_4^2$
$[t; t]$	—	$A \Psi_2^{(t+1)/3}$	—	—

† In a tetrad such that $\Psi_3 = 0$.

then by the same method which will be used to establish the $[t; t]$ solution in a type D background (equations (4.7), (4.8), (4.9) combined with 2.15) it can be shown that the only solution is

$$\Phi = A\Psi_3 \tag{4.2}$$

with $t = 3$ and A an arbitrary constant. $\Phi_{AB\ldots D}$ is simply a multiple of Ψ_{ABCD}.

If ι^A is not parallel to the remaining PND of Ψ_{ABCD} a more elaborate argument is required. Make a null rotation of the form (3.7) such that μ^A is parallel to this PND. Then the ZRM field equations are of the form (3.8). Define quantities ϕ and θ by

$$\begin{aligned} \phi &= \Phi\Psi_3^{-1/2} \\ \theta &= a\phi(t+1)^{-1} \end{aligned} \tag{4.3}$$

and in component form (3.8) becomes

$$þ\phi = \rho\phi \tag{4.4a}$$

$$ð\phi = \tau\phi \tag{4.4b}$$

$$þ\theta = ð'\phi - (t-1)\tau'\phi \tag{4.5}$$

$$ð\theta = þ'\phi - (t-1)\rho'\phi \tag{4.6}$$

A further factorization of ϕ by $\Psi_3^{1/2}$ reduces (4.4) to a form in which the lemma proved in the Appendix may be applied, hence (4.4) admits two functionally independent solutions. These then generate inhomogeneous terms in equations (4.5) and (4.6). Calculation of the commutator $[þ, ð]$ on θ shows that the integrability conditions on (4.5), and (4.6) are satisfied identically, hence the complete solution of (4.4) to (4.6) consists of two functionally independent particular integrals plus solutions of the homogeneous equations, which are again seen to exist by our lemma.

In addition, solutions to the homogeneous equations are associated with null (type $[t + 1; 0]$) ZRM test fields (see Table 1). Therefore in a type III space-time with a ZRM test field as described above there are exactly two functionally independent solutions of (4.4) to (4.6) modulo type $[t + 1; 0]$ test fields.

Now suppose that the background is type D and write

$$\Phi = A\Psi_2^{(t+1)/3}. \tag{4.7}$$

Since $t = r$ equations (3.1a–d) reduce to

$$þA = ðA = þ'A = ð'A = 0 \tag{4.8}$$

A is, however, of GHP type (0, 0) hence combination of (4.8) with (2.3) and recollection of the defining properties of a null tetrad yields

$$\nabla_a A = 0 \tag{4.9}$$

which establishes the result.

The remaining assertations of Table 1 are demonstrated in like manner, choosing ι^A to be such that $\Psi_3 = 0 (\Psi_2 \neq 0)$ or $\Psi_4 = 0$ $(\Psi_2 = 0, \Psi_3 \neq 0)$, removal of the indicated factor and use of Lemma 1.

The function ζ is essentially a gauge function. In the GHP formalism this is evident since it is ζ which carries the GHP type of the test field. For example, a type $[t; 0]$ field has GHP type $(t, 0)$ while Ψ_2 and Ψ_3 have respectively GHP type $(0, 0)$ and $(2, 0)$, and ζ has GHP type $(1, 0)$. Thus the GHP types of $\zeta^t \Psi_2^{1/3}$ and $\zeta^{t-1}\Psi_3^{1/2}$ are respectively $t(1, 0) + \frac{1}{3}(0, 0) = (t, 0)$ and $(t-1)(1, 0) + \frac{1}{2}(2, 0) = (t, 0)$ as required.

Further, Table 1 remains valid for neutrino fields (type $[1, 0]$), hence all null ZRM test fields in algebraically special space-times may be constructed from neutrino fields propagating along the repeated PND of Ψ_{ABCD}.

Finally it is possible to conclude that in an algebraically general space-time any ZRM test field must have the maximum number of PND's, for if it possesses a PND of multiplicity more than one, choose o^A to be parallel to this direction. Then regardless of the choice of ι^A, the analysis will yield equations similar to (3.1e). These in turn imply, via the Goldberg–Sachs theorem, that the space is algebraically special, which is contrary to the original assumption. Therefore each PND of the test field must be of multiplicity one.

It should be noted that nothing has been said concerning the existence of general neutrino fields or Maxwell fields of type $[1; 1]$ and there is a no reason to suspect that any algebraic restrictions exist in association with their appearance in algebraically special space-times.

Appendix

Lemma. If for some value of p, q there exist functions A, B such that $[þ, ð] = Aþ + Bð$, then there exist two functionally independent solutions of the equations

$$þZ = ðZ = 0 \tag{A.1}$$

where z has GHP type (p, q).

This result holds, *mutatis mutandis*, for any pair of $þ, þ', ð, ð'$.

Proof: Given $[þ, ð] = Aþ + Bð$, solve the pair of first-order homogeneous equations

$$þC = BC; \qquad ðD = -AD \tag{A.2}$$

for C and D. For a (p, q) object $þ \equiv l^a \nabla_a + G$ and $ð = m^a \nabla_a + H$. (The exact forms of G and H being unimportant for what is to follow.) Define

$$\tilde{þ} = \frac{1}{D} þ = \tilde{l}^a \nabla_a + \tilde{G} \tag{A.3a}$$

$$\tilde{ð} = \frac{1}{C} ð = \tilde{m}^a \nabla_a + \tilde{H} \tag{A.3b}$$

It is easy to show that $[\text{Þ}, \eth]\eta = 0$ for all η, i.e.

$$[\tilde{l}^a\nabla_a, \tilde{m}^b\nabla_b] = 0 \tag{A.4a}$$

$$\tilde{l}^a\nabla_a\tilde{H} = \tilde{m}^a\nabla_a\tilde{G} \tag{A.4b}$$

There are, therefore, two functionally independent solutions $\zeta_i (i = 1, 2)$ of

$$\tilde{l}^a\nabla_a\zeta_i = \tilde{m}^a\nabla a\zeta_i = 0 \tag{A.5}$$

Choosing a coordinate system such that $l^a = \delta^a o$, define

$$Z_i = \zeta_i \exp\left[-\int\tilde{G}\,dx^0\right] \tag{A.6}$$

Then

$$\tilde{\text{Þ}} Z_i = (\tilde{l}^a\nabla_a + \tilde{G})Z_i = 0 \tag{A.7a}$$

Similarly

$$\tilde{\eth}Z_i = 0 \tag{A.7b}$$

But

$$\tilde{\text{Þ}}Z_i = \tilde{\eth}Z_i = 0 \Rightarrow \text{Þ}Z_i = \eth Z_i = 0 \tag{A.8}$$

and the lemma is proved.

Acknowledgements

One of us (B.V.) would like to thank the other (A.H.) for finding all the references.

References

Bell, P. and Szekeres, P. (1972). *International Journal of Theoretical Physics*, Vol. 6, No. 2, p. 111.

Collinson, C. D. (1973). *J. Gen. Rel. Grav.*, **4**, 211.

Geroch, R. Held, A. and Penrose, R. (1973). *Journal of Mathematical Physics*, **14**, 874.

Goldberg, J. N. and Sachs, R. (1962). *Acta Physica Polonica*, **22**, Suppl., 13.

Newman, E. T. and Penrose, R. (1962). *Journal of Mathematical Physics*, **3**, 566.

Penrose, R. (1965). *Proceedings of the Royal Society*, **284A**, 159.

Pirani, F. A. E. (1965). Brandeis Summer Institute, 1964. Prentice-Hall, Englewood Cliffs, N.J.

Robinson, I. (1961). *Journal of Mathematical Physics*, **2**, 290.